

Asymptotic laws for nonconservative self-similar fragmentations

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Abstract We consider a self-similar fragmentation process in which the generic particle of size x is replaced at probability rate x^α by its offspring made of smaller particles, where α is some positive parameter. The total of offspring sizes may be both larger or smaller than x with positive probability. We show that under certain conditions the typical size in the ensemble is of the order $t^{-1/\alpha}$ and that the empirical distribution of sizes converges to a random limit which we characterise in terms of the reproduction law.

1 Introduction

We study the following continuous-time model of particle fragmentation. Each particle in ensemble is characterised by a positive quantity which we call *size*. The generic particle of size x lives a random exponentially distributed time with parameter x^α , where α is some fixed real number. During the lifetime the size does not vary and at the end the particle splits into fragments of sizes $x\xi_j$, where $\{\xi_j\}$ is independent of the lifetime of the particle and follows a given probability distribution called *reproduction law*. Each particle is autonomous, meaning that the splitting probability rate and the descendant fragment sizes depend only on the size of the particle and not on the history of this or other coexisting particles. We are interested in the case $\alpha > 0$, when particles with smaller size tend to live longer. We refer to the proceedings [14] and the survey [1] for a number of examples arising in physics and chemistry.

The idea of the model was suggested by Kolmogorov in [22], and the first results are due to Filippov [19]. Brennan and Durrett [17, 18] rediscovered an instance of the model in the context of binary interval splitting. In a recent series of papers Bertoin [6, 7, 8] introduced more involved fragmentation processes in which a particle may produce infinitely many generations within an arbitrary time period or may undergo a continuous size erosion; see also [2, 5, 9, 29, 30, 35] for related examples.

The research so far was mainly focussed on the *conservative* case $\sum \xi_j = 1$ when the total size is preserved by each splitting. It has been shown that the particles demonstrate quite a regular long-run behaviour: the typical size in the ensemble is of the order $t^{-1/\alpha}$ and a scaled empirical distribution of sizes converges to a nonrandom limit. Filippov [19] proved the convergence of empirical distributions in probability (see also [8]), while Brennan and Durrett [18] showed convergence with probability one in the binary case. Baryshnikov and Gnedin [3] studied a sequential interval packing problem which may be seen as a binary instance of *dissipative* fragmentation with $\sum \xi_j \leq 1$ and $\mathbb{P}(\sum \xi_j < 1) > 0$, and proved convergence of the mean measures associated with the empirical distributions. We mention that some special cases of these mathematical results have also appeared in the literature in physics, see e.g. [4, 23] and references therein.

Conservative or dissipative fragmentations can be treated both as continuous-time interval splitting schemes, similar to discrete-time random recursive constructions (as in [28]), or as state-discretised processes with values in Kingman's partition structures [6, 7, 8, 9, 11]. These approaches fail completely if the reproduction law allows the possibility of size *creation*, when the total size of the offspring may exceed the size of the parent particle. Such 'improper fragmentations' are both physically plausible and useful in the situations where the generalised size models some nonadditive quantity like, e.g. surface energy by aerosols.

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By allocating particle of size x at $-\log x$ the fragmentation process can be seen as a branching random walk with location-dependent sojourn times. From this viewpoint, a constraint on the sum of sizes seems rather odd, which suggests that such a condition is not essential for the asymptotics. In this paper we argue that this intuition is indeed correct, in the sense that the convergence of properly scaled empirical distributions of sizes holds under fairly general assumptions on the reproduction law. Though we do require that the individual offspring sizes cannot exceed the parent size, there is no constraint on the total offspring size. A new feature appearing in the general nonconservative case is that the limit of scaled empirical distributions is not completely deterministic, rather involves a random factor which admits a characterisation by a distributional fixed-point equation. This phenomenon reminds us, of course, of strong limit theorems for branching random walks; we shall discuss the connection and the differences in Section 4.2.

The rest of this work is organised as follows. Notation and basic assumptions are given in Section 2.1, and Section 2.2 introduces the genealogical structure and the so-called intrinsic martingale which plays a fundamental role in branching processes. Then we compute the first moment of power sums and then determine its asymptotics using a contour integral. This yields the convergence of mean measures in Section 3.3. An alternative approach based on a limit theorem of Brennan and Durrett is presented in Section 3.4. The main result of convergence of scaled empirical measures is proved in Section 4.1. Then we provide some examples in Section 5, and finally, in Section 6, we sketch the extension of the preceding results to self-similar fragmentations with possibly infinite reproduction measure.

2 Preliminaries

2.1 Definitions and assumptions on the reproduction law

The collection $\{\xi_j\}_{j \in \mathbb{N}}$ of offspring sizes of a unit particle is identified with a random sequence of non-negative real numbers ranked in the decreasing order (by convention, $\xi_j = 0$ when j is larger than the number of children). We shall also view $\{\xi_j\}_{j \in \mathbb{N}}$ as a random set, defined formally as a counting random measure $\sum \delta_{\xi_j}$ on $]0, 1]$. Basically we require that

$$\xi_j \in [0, 1], \quad \mathbb{E} \# \{j : \xi_j > 0\} > 1, \quad \mathbb{P}(\xi_1 = 1) < 1. \quad (1)$$

Many features of the fragmentation process can be expressed in terms of the *structural measure*

$$\sigma(B) = \mathbb{E} \# \{j : \xi_j \in B\}, \quad B \subset]0, 1],$$

and its Mellin transform

$$\phi(\beta) = \int_0^1 x^\beta \sigma(dx) = \mathbb{E} \sum_{j=1}^{\infty} \xi_j^\beta$$

which we call the *characteristic function*. In particular, the conditions in (1) amount to the assumptions that σ is supported by $[0, 1]$, that $\sigma]0, 1] > 1$ and that $\sigma\{1\} < 1$.

Because $|\phi(\beta)| \leq \phi(\Re \beta)$, the natural domain of definition of ϕ is a complex halfplane to the right of the convergence abscissa β_a of the integral. If $\beta_a = -\infty$ the halfplane is the whole plane, and otherwise the halfplane may be open or closed. The characteristic function is analytical in the halfplane, strictly decreasing on the real axis, and in view of (1) satisfies $\phi(0) > 1$ and $\phi(\beta) \rightarrow \sigma\{1\} < 1$ as $\Re \beta \rightarrow \infty$.

It is crucial for our results and will be assumed throughout that there exists the *Malthusian exponent* $\beta^* > 0$ satisfying the equation

$$\phi(\beta) = 1. \quad (2)$$

If the Malthusian exponent exists then it is unique and there are no solutions to (2) in the halfplane $\Re \beta > \beta^*$. And if some $\beta \neq \beta^*$ with $\Re \beta = \beta^*$ satisfies (2) then σ is *arithmetic*, meaning that σ is a discrete measure supported by a geometric sequence. Note that in the conservative case $\beta^* = 1$ and in the dissipative case $\beta^* < 1$.

Equation (2) has no real solutions (thus the Malthusian exponent is not defined) only if $\phi(\beta_a+) < 1$. An example of this situation is the measure

$$\sigma(dx) = c \mathbf{1}_{\{x < 1/2\}} x^{-3/2} \log^{-2} x \, dx$$

with a suitable choice of c , and $\beta_a = 1/2$.

Further assumptions about the reproduction law will be introduced in a due place. Specifically, the L^2 -convergence results in Sections 2.2 and 4.1 require that

$$\mathbb{E} \left(\sum_{j=1}^{\infty} \xi_j^{\beta^*} \right)^2 < \infty. \quad (3)$$

Example. Consider a dissipative reproduction law induced by the uniform stick-breaking. Let U_0, U_1, \dots be i.i.d. uniform, and let $\{\xi_j\}_{j \in \mathbb{N}}$ stand for the rearrangement in the decreasing order of the sequence

$$(1 - U_j) \prod_{k=0}^{j-1} U_k, \quad j = 1, 2, \dots$$

meaning that the uniform portion of the size $1 - U_0$ is lost, and the rest is fragmented by the ‘random alms’ principle (as Halmos called stick-breaking). Equivalently, the offspring $\{\xi_j\}_{j \in \mathbb{N}}$ of a unit particle can be seen as the collection of sizes of a random Poisson-Dirichlet distribution with parameter 1, upon removing a size selected by a size-biased pick. Building the characteristic function

$$\phi(\beta) = \sum_{j=1}^{\infty} \frac{1}{(1 + \beta)^{j+1}} = \frac{1}{\beta(\beta + 1)}$$

we see that the abscissa is at $\beta_a = 0$ and the Malthusian exponent is $\beta^* = (-1 + \sqrt{5})/2$.

It should be noted that there is no substantial constraint on σ imposed by the requirement that σ be a structural measure. Given σ on $[0, 1]$, satisfying $\sigma[0, 1] > 1$ and $\sigma\{1\} < 1$, a possible reproduction law satisfying (1) with this structural measure can be constructed as follows. Decompose $\sigma = \sigma_1 + \sigma_2$ so that σ_1 be probability measure and σ_2 some other measure. Let ξ_1 be a random point with distribution σ_1 and let ξ_2, ξ_3, \dots be the atoms of a Poisson point process on the unit interval, with intensity measure σ_2 . Clearly the point process $\{\xi_j\}$ will have intensity σ .

2.2 Genealogical structure and the intrinsic martingale

In this section, we develop some elements on the genealogical structure of the fragmentation. This can be viewed as a different parametrisation of the process, in which the natural time is replaced by the generation of the different particles. Specifically, we consider the infinite tree

$$\mathcal{U} := \bigcup_{n=0}^{\infty} \mathbb{N}^n,$$

with the convention $\mathbb{N}^0 = \{\emptyset\}$. The elements of \mathcal{U} are called nodes. For each $u = (u_1, \dots, u_n) \in \mathcal{U}$, we call n the *generation* of u and write $|u| = n$, with the obvious convention $|\emptyset| = 0$. When $n \geq 1$ and $u = (u_1, \dots, u_n)$, we call $u^- := (u_1, \dots, u_{n-1})$ the father of u , and $u, i = (u_1, \dots, u_n, i)$ the i -th child of u .

It will be assumed that $\alpha > 0$ and that the fragmentation process X starts from a single particle with unit size, unless explicitly indicated. We encode X by putting marks on the nodes of the infinite tree \mathcal{U} as follows. The initial particle with unit size corresponds to the ancestor \emptyset , and the mark of \emptyset is the triple $(\xi_\emptyset, g_\emptyset, d_\emptyset) = (1, 0, \mathbf{e})$ where \mathbf{e} is the instant of the first reproduction, so that $[g_\emptyset, d_\emptyset]$ is the time-interval during which the ancestor particle is alive. The nodes of the tree at the first generation are used as the labels of the particles arising at the first split, i.e. $\xi_1, \dots, \xi_j, \dots$. Again, the mark associated to each of these nodes $i \in \mathbb{N}^1$, is the triple (ξ_i, g_i, d_i) , where $g_i = d_\emptyset$ stands for the birth-time of the i -th child of the ancestor, and $d_i = g_i + \xi_i^\alpha \mathbf{e}_i$ for its death-time. And we iterate the same construction with each particle at each generation.

Plainly, the description of the dynamics of fragmentation entails that its representation as a random marked tree enjoys the branching property. Specifically, the distribution of the random marks can be

described recursively as follows: Given the marks $((\xi_v, g_v, d_v), |v| \leq n)$ at nodes of the first n generations, the marks at nodes of generation $n+1$ can be expressed in the form

$$(\xi_u, g_u, d_u) = (\xi_{u-} \tilde{\xi}_u, d_{u-}, d_{u-} + \xi_u^\alpha \mathbf{e}_u), \quad u = (u_1, \dots, u_{n+1}),$$

where $u- = (u_1, \dots, u_n)$ is the father of the node u , and

- when $u- = (u_1, \dots, u_n) \in \mathbb{N}^n$ describes the nodes at the n -th generation, $(\tilde{\xi}_{u_1, \dots, u_n, i}, i \in \mathbb{N})$ are i.i.d. random sequences distributed according to the reproduction law,
- when u describes the nodes at the $(n+1)$ -th generation, the variables \mathbf{e}_u are i.i.d. standard exponential variables which are independent of the sequences $\tilde{\xi}_{u-, \dots}$.

One says that extinction occurs when $\xi_u = 0$ for all nodes u at generation n for some large enough $n \in \mathbb{N}$. The assumptions (1) ensure that the process $Z_n := \#\{u \in \mathbb{N}^n : \xi_u > 0\}$ is a supercritical Galton-Watson process (possibly taking the value ∞), so the probability of non-extinction is strictly positive.

It is well-known from the works of Jagers [21], Nerman [31] and many other authors, that the Malthusian hypothesis for branching processes is connected to a remarkable martingale. The latter is often referred to as the *intrinsic martingale*, it plays a crucial role in the analysis of the asymptotic behaviour of branching processes. The following statement, expressed in terms of the genealogical coding, is part of the folklore (we refer to [33] for the last part of the claim). Recall that β^* denotes the Malthusian exponent.

Proposition 1 *Under assumptions (1) and (3), the process*

$$M_n := \sum_{|u|=n} \xi_u^{\beta^*}, \quad n \in \mathbb{N}$$

is a martingale which is bounded in L^2 and in particular, uniformly integrable. Its terminal value M_∞ is strictly positive conditionally on non-extinction, and satisfies the distributional identity

$$M_\infty \stackrel{d}{=} \sum_{j=1}^{\infty} \xi_j^{\beta^*} M_\infty^{(j)} \quad (4)$$

where $M_\infty^{(j)}$ are independent copies of M_∞ , also independent of $\{\xi_j\}_{j \in \mathbb{N}}$. The identity taken together with conditions (1), (3) and $\mathbb{E}M_\infty = 1$ characterises M_∞ uniquely.

Remark. Observe that in the important case when the reproduction law is conservative, in the sense that $\sum_{i=1}^{\infty} \xi_i = 1$ a.s., we have $\beta^* = 1$ and $M_n = 1$ for all $n \in \mathbb{N}$, so that the statement is trivial.

So far we have described the fragmentation process in terms of its genealogy; however the problems of interest are often expressed in terms of the natural time scale. More precisely, the configuration of the fragmentation process at time $t \geq 0$ consists in the set $\{X_j(t)\}_{j \in \mathbb{N}}$ of particles coexisting at time $t \geq 0$, that is in terms of random point measures

$$\sum \delta_{X_j(t)} = \sum_{u \in \mathcal{U}} \mathbf{1}_{\{g_u \leq t < d_u\}} \delta_{\xi_u}.$$

Recall we assume that the process starts with a sole particle of unit size, that is $X(0) = \{1, 0, \dots\}$. Denoting $X^{(y)}$ the fragmentation process that starts with a particle of size $y > 0$, we have the fundamental self-similarity identity

$$X^{(y)}(t) \stackrel{d}{=} y X(ty^\alpha). \quad (5)$$

The intrinsic martingale is indexed by the generations of the infinite tree. In terms of the time scale of the fragmentation process, we write $(\mathcal{F}_t)_{t \geq 0}$ for the natural filtration generated by the fragmentation process $X = (X(t), t \geq 0)$. Proposition 1 then yields the following result which could also be proved by techniques of so-called stopping lines applied to the branching process induced by the genealogical coding (see e.g. Theorem 6.3 and Corollary 6.6 in Jagers [21]).

Corollary 2 *Under the assumptions of Proposition 1 for every $t \geq 0$, we have*

$$\mathbb{E}(M_\infty \mid \mathcal{F}_t) = M(t, \beta^*) := \sum_j X_j^{\beta^*}(t),$$

hence the process $M(\cdot, \beta^)$ is a square-integrable (\mathcal{F}_t) -martingale with terminal value M_∞ .*

Proof. We know that M_n converges in $L^2(\mathbb{P})$ to M_∞ as n tends to ∞ , so

$$\mathbb{E}(M_\infty \mid \mathcal{F}_t) = \lim_{n \rightarrow \infty} \mathbb{E}(M_n \mid \mathcal{F}_t).$$

On the other hand, it is easy to deduce from the Markov property applied at time t that

$$\mathbb{E}(M_n \mid \mathcal{F}_t) = \sum_{i=1}^{\infty} X_i^{\beta^*}(t) \mathbf{1}_{\{G(X_i(t)) \leq n\}} + \sum_{|u|=n} \xi_u^{\beta^*} \mathbf{1}_{\{d_u < t\}},$$

where $G(x)$ stands for the generation of the particle x (i.e. $G(\xi_u) = |u|$), and d_u for the instant when the particle corresponding to the vertex u splits. However, for each fixed vertex $u \in \mathcal{U}$, d_u is bounded from below by the sum of $|u| + 1$ independent exponential variables which are independent of ξ_u . It follows that

$$\lim_{n \rightarrow \infty} \mathbb{E} \sum_{|u|=n} \xi_u^{\beta^*} \mathbf{1}_{\{d_u < t\}} = 0,$$

and we conclude that $\mathbb{E}(M_\infty \mid \mathcal{F}_t) = M(t, \beta^*)$. □

3 Asymptotics in mean

3.1 Power sums and their means

In the sequel, we shall find useful to consider the power-sum functionals

$$M(t, \beta) = \sum_j X_j^\beta(t)$$

and their means

$$m(t, \beta) = \mathbb{E}M(t, \beta).$$

For shorthand we sometimes refer to the size of a particle raised to the power β as the β -size, thus $M(t, \beta)$ is the total β -size of the population existing at time t . Two instances with obvious physical interpretations are the 0-size equal to the number of particles, and the 1-size equal to the total size of the ensemble.

As the set of particles alive at time t is a part of the set of particles that are born before time t , there is the inequality

$$m(t, \beta) = \mathbb{E} \sum_{i=1}^{\infty} X_i^\beta(t) \leq \mathbb{E} \sum_{u \in \mathcal{U}} \xi_u^\beta \mathbf{1}_{\{g_u \leq t\}},$$

where g_u denotes the birth-time of the particle labelled by the node u . Observe that for each node u at generation n , $g(u)$ can be bounded from below by the sum of n independent standard exponential variables which are also independent of ξ_u , and that, in the notation of Section 2.1,

$$\mathbb{E} \sum_{|u|=n} \xi_u^\beta = \phi(\beta)^n, \quad \beta > \beta_a, n \in \mathbb{N}.$$

It now follows from classical large deviation estimates that $m(t, \beta) < \infty$ for every $t \geq 0$ whenever $\beta > \beta_a$.

The first-split decomposition with application of (5) shows that $M(t, \beta)$ satisfies the distributional identity

$$M(t, \beta) \stackrel{d}{=} \mathbf{1}_{\{t < d_\emptyset\}} + \mathbf{1}_{\{t \geq d_\emptyset\}} \sum_j \xi_j^\beta M_j(\xi_j^\alpha(t - d_\emptyset)) \quad (6)$$

where d_\emptyset is the exponential life-time of the progenitor, and the M_j 's are independent replicas of $M(\cdot, \beta)$, which are also independent of d_\emptyset and $\{\xi_j\}$. Computing expectations we arrive at the integral equation

$$m(t, \beta) = e^{-t} + \int_0^t e^{-s} \int_0^1 m((t-s)x^\alpha, \beta) x^\beta \sigma(dx).$$

Differentiating we see that $m(\cdot, \beta)$ is a solution to the Cauchy problem for the integro-differential equation

$$\partial_t m(t, \beta) = -m(t, \beta) + \int_0^1 m(x^\alpha t, \beta) x^\beta \sigma(dx). \quad (7)$$

which must be complemented by the initial value $m(0, \beta) = 1$. Uniqueness of C^∞ solutions for equations of this type is shown in [20].

Equation (7) defines functions m for all $\Re \beta > \beta_a$. For the higher derivatives we have

$$\partial_t^k m(t, \beta) = \partial_t^k m(0, \beta) m(t, k\beta + \alpha),$$

thus m is increasing in t for $\beta < \beta^*$ and decreasing for $\beta > \beta^*$.

Solving (7) in power series is straightforward. Introducing

$$\psi(\beta) = 1 - \phi(\beta)$$

and then

$$\gamma(n, \beta) = \prod_{k=0}^{n-1} \psi(\beta + \alpha k) \quad (8)$$

(by convention, $\gamma(0, \beta) = 1$), we compute

$$m(t, \beta) = \sum_{n=0}^{\infty} \frac{(-t)^n}{n!} \gamma(n, \beta) \quad (9)$$

which is an entire function of $t \in \mathbb{C}$. It is indeed the right solution because from the formula for derivatives it is clear that $m(\cdot, \beta)$ should be C^∞ for $t \geq 0$. See [20] for yet another interesting representation of $m(t, \beta)$, in the form of a generalised Dirichlet series.

3.2 A contour integral

When $\phi(\beta)$ is a rational function, splitting $\gamma(\cdot, \beta)$ in linear factors shows that the series (9) represents a generalised hypergeometric function, in which case the $|t| \rightarrow \infty$ asymptotic expansions have been thoroughly studied by Mellin-Barnes' contour integral technique, see [27]. This method extends to the more general situation considered here (also see [3]).

Call β *singular* if $\psi(\beta + \alpha n) = 0$ for some integer $n \geq 0$. For singular β the series $m(t, \beta)$ is a polynomial, thus $m(t, \beta) = \gamma(n, \beta) t^n + O(t^{n-1})$ for $t \rightarrow \infty$. Analogous asymptotics hold also for nonsingular β , but it is more difficult to justify, because m is then an infinite series which starts alternating from some term. A good heuristic amounts to substituting $m \sim ct^a$ into (7) – the left-hand side is then of the order t^{a-1} while the right-hand side is $o(t^a)$ exactly when $\psi(\beta + \alpha a) = 0$, which suggests that $a = (\beta^* - \beta)/\alpha$ is the right exponent. Although this kind of reasoning can be made precise it gives no idea of the coefficient, see [20].

Assuming β nonsingular we extrapolate the function $\gamma(\cdot, \beta)$ from the integer values to arbitrary complex values z (such that $\phi(\alpha z + \beta)$ is defined) by means of the formula

$$\gamma(z, \beta) = \prod_{k=0}^{\infty} \frac{\psi(\beta + \alpha k)}{\psi(\beta + \alpha(k+z))}. \quad (10)$$

Convergence of the product follows as in [3], Section 5. Thus defined, γ satisfies the functional equation

$$\gamma(z+1, \beta) = \psi(\beta + \alpha z) \gamma(z, \beta)$$

reminiscent of the well-known equation for Euler's gamma function.

All singularities of the function $\gamma(\cdot, \beta)$ are the poles located at roots of (2). Let

$$\mathcal{P}_\beta = \{z : \exists n \geq 0, \psi(\beta + \alpha(n + z)) = 0\}$$

be the set of singular points. Because (2) has no solutions to the right of β^* , the rightmost point of \mathcal{P}_β is $z_\beta := (\beta^* - \beta)/\alpha$, where $\gamma(\cdot, \beta)$ has a simple pole provided $\beta^* > \beta_a$.

Still assuming β nonsingular we have $\mathcal{P}_\beta \cap \{0, 1, \dots\} = \emptyset$. Since the poles of $\Gamma(-z)$ are nonnegative integers, the function $\Gamma(-z)\gamma(z, \beta)t^z$ (with t as parameter) also has these poles, with residue $(-1)^n t^n \gamma(n, \beta)/n!$ at $z = n$. Defining \mathcal{C} to be a vertical line between $\Re z_\beta$ and $n_\beta := \min(0, \lceil \Re z_\beta \rceil)$ we obtain by the residue theorem and an estimate of γ

$$\sum_{n=n_\beta}^{\infty} \frac{(-t)^n}{n!} \gamma(n, \beta) = \frac{1}{2\pi i} \int_{\mathcal{C}} \Gamma(-z) \gamma(z, \beta) z^t dz. \quad (11)$$

If $\beta^* > \beta_a$ the function $\gamma(\cdot, \beta)$ is meromorphic in an open strip containing the line $\Re z = z_\beta$, and the residue at z_β is

$$\text{Res}_{z_\beta} \gamma(z, \beta) = \frac{\psi(\beta)}{\alpha \psi'(\beta^*)} \gamma\left(\frac{\beta^* - \beta}{\alpha}, \alpha + \beta\right)$$

as it follows from the identity

$$\gamma(s, \beta) = \frac{\psi(\beta)}{\psi(\beta + \alpha s)} \gamma(s, \alpha + \beta)$$

upon expanding the ratio. Replacing \mathcal{C} by another integration contour \mathcal{C}' located in the half-plane $\Re z < \Re z_\beta$ so that all poles of $\gamma(\cdot, \beta)$ in this half-plane lie to the left of \mathcal{C}' we obtain the principal-term asymptotics of m .

Theorem 3 *Suppose $\beta^* > \beta_a$, the structural measure σ is nonarithmetic and conditions (1) hold, then*

$$m(t, \beta) \sim \Gamma\left(\frac{\beta - \beta^*}{\alpha}\right) \frac{\psi(\beta)}{\alpha \psi'(\beta^*)} \gamma\left(\frac{\beta^* - \beta}{\alpha}, \alpha + \beta\right) t^{(\beta^* - \beta)/\alpha}, \quad \text{as } t \rightarrow \infty. \quad (12)$$

for $\Re \beta > \beta_a$.

Using the identity $\gamma(-z, \alpha z + \beta) \gamma(z, \beta) = 1$ we can re-write the γ -factor in (12) as

$$\gamma\left(\frac{\beta^* - \beta}{\alpha}, \alpha + \beta\right) = \frac{1}{\gamma((\beta - \beta^*)/\alpha, \alpha + \beta^*)}.$$

The restriction of ψ to the real segment $]\beta_a, \infty[$ is plainly a concave increasing function, so the condition $\beta^* > \beta_a$ entails $0 < \psi'(\beta^*) < \infty$. We also remark that in the arithmetic case other poles on the line $\Re z = \beta^*$ would contribute to the coefficient. Further terms of the asymptotic expansion can be obtained by pulling the integration contour through other poles left of β^* , as long as ϕ admits a meromorphic continuation, which can go beyond the convergence abscissa. If β^* is the only point of \mathcal{P}_β in a closed strip $\beta^* - \theta \leq \Re z \leq \beta^*$ then the rest-term in (12) is estimated as $O(t^{(\beta^* - \beta)/\alpha - \epsilon})$ with $\epsilon = \min(1, \theta/\alpha)$.

3.3 Convergence of the mean measures

We encode the configuration of sizes $X(t) = \{X_j(t)\}$ into the random measure

$$\sum_j X_j^{\beta^*}(t) \delta_{t^{1/\alpha} X_j(t)}.$$

The associated mean measure σ_t^* is defined by the formula

$$\int_0^\infty f(x) \sigma_t^*(dx) = \mathbb{E} \sum_j f(t^{1/\alpha} X_j(t)) X_j^{\beta^*}(t) \quad (13)$$

which is required to hold for all compactly supported continuous functions f . It is easily seen that σ_t^* is a probability measure. Our next goal is to show that the measures σ_t^* converge weakly to a probability measure ρ on $]0, \infty[$.

Because

$$t^{(\beta-\beta^*)/\alpha} m(t, \beta) = \int_0^\infty x^{\beta-\beta^*} \sigma_t^*(dx)$$

the convergence of $t^{(\beta-\beta^*)/\alpha} m(t, \beta)$ implied by (12) amounts to the convergence of power moments

$$\int_0^\infty x^{\beta-\beta^*} \sigma_t^*(dx) \rightarrow \Gamma\left(\frac{\beta-\beta^*}{\alpha}\right) \frac{\psi(\beta)}{\alpha\psi'(\beta^*)} \frac{1}{\gamma((\beta-\beta^*)/\alpha, \alpha+\beta^*)}.$$

Specialising $\beta = \beta^* + \alpha k$ this simplifies to

$$\int_0^1 x^{k\alpha} \sigma_t^*(dx) \rightarrow \frac{(k-1)!}{\alpha\psi'(\beta^*)} \prod_{j=1}^{k-1} \frac{1}{\psi(\beta^* + \alpha j)}$$

in application of (10). It is easy to check and is well known [13, 18] that for $n = 0, 1, \dots$ the related moment problem is determinate, whence the following result.

Theorem 4 *Under assumptions of Theorem 3 the measures σ_t^* converge weakly, as $t \rightarrow \infty$, to a probability measure ρ uniquely determined by its power moments*

$$\int_0^\infty x^{k\alpha} \rho(dx) = \frac{(k-1)!}{\alpha\psi'(\beta^*)} \prod_{j=1}^{k-1} \frac{1}{\psi(\beta^* + \alpha j)}, \quad k = 1, 2, \dots \quad (14)$$

Theorems 3 and 4 imply that

$$t^{(\beta-\beta^*)/\alpha} m(t, \beta) \rightarrow \int_0^\infty x^{\beta-\beta^*} \rho(dx) \quad (15)$$

which extends the convergence of expectations in (13) to a wider class of functions f .

3.4 Self-similar stick-breaking process

A key tool in the conservative case treated in [17, 18, 6, 7, 8] has been the following observation related, somewhat paradoxically, to the simplest dissipative case of a singleton ensemble with reproduction law $\{\eta\}$, where η is a random variable assuming values in $]0, 1]$. That is to say, if at some time the particle has size x then, independently of the history, the particle shrinks with probability rate x^α and the new size after the shrink becomes $x\eta$ where η follows $\hat{\sigma}$, and $\hat{\sigma}$ is the probability law of η .

We recollect briefly a result from [18] (also see [8, 10]). Introduce

$$\hat{\psi}(\beta) = 1 - \int_0^1 x^\beta \hat{\sigma}(dx)$$

and suppose $\hat{\psi}(0+) < \infty$. Let L_t be the sole size at time t , and $\hat{m}(t, \beta) = \mathbb{E} L_t^\beta$. Assuming $\hat{\sigma}$ non-arithmetic, Brennan and Durrett [18] proved that for $t \rightarrow \infty$

$$t^{1/\alpha} L_t \xrightarrow{d} Y^{1/\alpha} \quad \text{and} \quad \hat{m}(t, \beta) \rightarrow \mathbb{E} Y^{\beta/\alpha}$$

where Y is a random variable with moments

$$\mathbb{E} Y^k = \frac{(k-1)!}{\alpha \widehat{\psi}'(0+)} \prod_{j=1}^{k-1} \frac{1}{\widehat{\psi}(\alpha j)}.$$

The convergence of moments $\widehat{m}(t, \beta)$ was shown for all real β strictly to the right of the convergence abscissa of $\widehat{\psi}$ (which is nonpositive due to the normalisation $\widehat{\sigma}]0, 1] = 1$). They also suggested the explicit representation

$$Y \stackrel{d}{=} \sum_{k=0}^{\infty} \epsilon_k \prod_{j=0}^k \eta_j^{\alpha} \quad (16)$$

where all η_j, ϵ_j are independent, ϵ_j is mean one exponential, η_j for $j > 0$ are replicas of η , and η_0 follows the law

$$\mathbb{P}(\eta_0 \in dx) = \frac{\widehat{\sigma}[x, 1]dx}{\widehat{\psi}'(0)x} \quad x \in]0, 1].$$

Each product in (16) corresponds to the size of the particle after k splits, conditionally given the initial size is η_0 , thus formula (16) identifies Y with the well-known exponential functional of a (stationary) compound Poisson process, see e.g. the survey [13].

In the conservative fragmentation case, the above ‘stick-breaking’ process describes the evolution of a particle tagged by an atom of isotope that was injected at a random uniform location into the progenitor unit size. The mechanism which determines the line of descent of the tagged particle amounts, at each consecutive split, to a random size-biased pick from the child particles. Thus, defining $\widehat{\sigma}(dx) := x \sigma(dx)$ to be the distribution of a size-biased pick from $\{\xi_j\}$, we obtain the relation $\widehat{m}(t, \beta - 1) = m(t, \beta)$ which was observed in [18], p. 112.

In the general nonconservative case choosing $\widehat{\sigma}(dx) := x^{\beta^*} \sigma(dx)$, we still get $\widehat{m}(t, \beta - \beta^*) = m(t, \beta)$ and $\widehat{\psi}(z) = \psi(z + \beta^*)$. An interpretation of these relations akin to the tagged fragment process is available through the so-called *spine* which appears in the conceptual approach of Lyons *et al.* [26, 25] for convergence of martingales in branching processes. The measure ρ may be identified with the distribution of $Y^{1/\alpha}$. A consequence of this discussion and the result of Brennan and Durrett is the following corollary.

Corollary 5 *The conclusion of Theorem 4 remains valid even if the assumption $\beta^* > \beta_a$ is replaced by the weaker $\psi'(\beta^*+) < \infty$*

The tiny improvement upon Theorem 4 appears in the case where the characteristic function is defined in a closed half-plane and $\beta^* = \beta_a$, i.e. the Malthusian exponent falls exactly on the convergence abscissa. An example of such situation is the structural measure of the form $\sigma(dx) = c \mathbf{1}_{\{x < 1/2\}} x^{-3/2} |\log x|^{-3} dx$ with a suitable c . The method based on contour integration requires in such cases the analytical continuation of ϕ in a domain to the left of β_a .

Alternatively, along the lines in [17, 18, 10], the renewal theory can be applied also in the nonconservative case, to prove first the convergence of measures $\sigma_t^* \rightarrow \rho$ and then to justify the asymptotics of mean power-sums using the uniform integrability.

4 Limit theorem for the empirical measure of the fragments

4.1 L^2 -convergence

Our principal result improves on the convergence of the mean measures $\sigma_t \rightarrow \rho$ in Theorem 4 and says that the scaled empirical measures induced by $X(t)$ converge in a L^2 -sense to the measure $M_\infty \rho$, where M_∞ is the terminal value of the intrinsic martingale (cf. Proposition 1).

Theorem 6 Assume (1), (3), that $\beta^* > \beta_a$ and that σ is nonarithmetic. Then for any bounded continuous f

$$L^2\text{-}\lim_{t \rightarrow \infty} \sum_j X_j^{\beta^*}(t) f(t^{1/\alpha} X_j(t)) = M_\infty \int_0^\infty f(x) \rho(dx).$$

Proof. We need to show that

$$\mathbb{E} \left(\sum_{i,j} X_i^{\beta^*}(t) f(t^{1/\alpha} X_i(t)) X_j^{\beta^*}(t) g(t^{1/\alpha} X_j(t)) \right) \rightarrow \mathbb{E} M_\infty^2 \left(\int_0^\infty f(x) \rho(dx) \right) \left(\int_0^\infty g(x) \rho(dx) \right) \quad (17)$$

for positive f and g bounded from above by 1. Indeed, suppose (17) is shown. Denote

$$A_t = \sum_j X_j^{\beta^*}(t) f(t^{1/\alpha} X_j(t)).$$

Take $f = g$ to conclude from (17) that

$$\lim_{t \rightarrow \infty} \mathbb{E} A_t^2 = \mathbb{E} M_\infty^2 \left(\int_0^\infty f(x) \rho(dx) \right)^2.$$

Similarly, setting $g = 1$

$$\lim_{t \rightarrow \infty} \mathbb{E}(A_t M(t, \beta^*)) = \mathbb{E} M_\infty^2 \int_0^\infty f(x) \rho(dx).$$

Recalling from Corollary 2 that $\mathbb{E} M(t, \beta^*)^2 \rightarrow \mathbb{E} M_\infty^2$ and combining the above we get the desired

$$\lim_{t \rightarrow \infty} \mathbb{E} \left(A_t - M(t, \beta^*) \int_0^\infty f(x) \rho(dx) \right)^2 = 0.$$

To prove (17) let us replace t by $t + s$ and condition on the configuration of sizes $X(s)$. At time $t + s$ two coexisting particles may stem from the same ancestor that lived at time s or from two different ancestors; write $i \sim_s j$ in the first case, and write $i \not\sim_s j$ in the second. The sum in the left-hand side of (17) is split then in two

$$S_1 + S_2 = \mathbb{E} \left(\sum_{i \sim_s j} \cdots | X(s) \right) + \mathbb{E} \left(\sum_{i \not\sim_s j} \cdots | X(s) \right).$$

Using the fundamental self-similarity relation (5) and the Markov nature of the fragmentation process we estimate the first sum as

$$S_1 \leq \sum_k X_k^{2\beta^*}(s) \mathbb{E} \left(\sum_j X_j^{\beta^*}(t) \right)^2$$

hence by (12) and Corollary 2

$$\mathbb{E} S_1 < \text{const } s^{-\beta^*/\alpha} \rightarrow 0 \quad \text{as } s \rightarrow \infty$$

uniformly in t .

Dealing with S_2 requires more effort. We use the parallel notation $y_j = X_j(s)$. Write $i \searrow k$ if the size $X_i(t + s)$ stems from y_k . By independence, the descendants of different particles with sizes y_k and y_ℓ evolve independently, thus grouping the sizes $X_j(t + s)$ by the ancestors at time s yields

$$S_2 = \sum_{k \neq \ell} \left(\mathbb{E} \sum_{i \searrow k} \cdots \right) \left(\mathbb{E} \sum_{j \searrow \ell} \cdots \right)$$

However, by self-similarity and convergence of the mean measures

$$\mathbb{E} \sum_{i \searrow k} y_k^{\beta^*} X_i^{\beta^*}(ty_k^\alpha) f((t+s)^{1/\alpha} y_k X_i(ty_k^\alpha)) \rightarrow y_k^{\beta^*} \int_0^\infty f(x) \rho(dx)$$

$$\mathbb{E} \sum_{j \searrow \ell} y_\ell^{\beta^*} X_j^{\beta^*}(ty_\ell^\alpha) g((t+s)^{1/\alpha} y_\ell X_j(ty_\ell^\alpha)) \rightarrow y_\ell^{\beta^*} \int_0^\infty g(x) \rho(dx)$$

as $t \rightarrow \infty$, therefore by dominated convergence

$$\mathbb{E} S_2 \sim \left(\int_0^\infty f(x) \rho(dx) \right) \left(\int_0^\infty g(x) \rho(dx) \right) \mathbb{E} \sum_{k \neq \ell} X_k^{\beta^*}(s) X_\ell^{\beta^*}(s)$$

as $s \rightarrow \infty$. It remains to note that

$$\mathbb{E} \sum_{k \neq \ell} X_k^{\beta^*}(s) X_\ell^{\beta^*}(s) \sim \mathbb{E} \sum_{k, \ell} X_k^{\beta^*}(s) X_\ell^{\beta^*}(s) = \mathbb{E} M_s^2 \rightarrow \mathbb{E} M_\infty^2$$

because

$$\mathbb{E} \sum_k X_k^{2\beta^*}(s) = m(t, 2\beta^*) \rightarrow 0.$$

□

Remarks. We mention that if we replace the assumption (3) by the weaker

$$\mathbb{E} \left(\sum_j \xi_j^{\beta^*} \right)^p < \infty$$

for some $1 < p \leq 2$, the calculation of the expectation of the sum of the p -th powers of jumps then shows that the intrinsic martingale M_n in Proposition 1 is bounded in L^p , see e.g. Neveu [32]. Then techniques of Nerman [31], based on generalisations of the law of large numbers for branching processes, can be adapted to extend Theorem 6 to this situation (of course, L^2 convergence then has to be replaced by L^p -convergence).

Finally, it is known that in the binary conservative case the scaled empirical measures converge with probability one [18]. It would be interesting to extend this result to the nonconservative case.

In parallel to (15) there is the following extension of Theorem 6 to power functions.

Corollary 7 *Under assumptions of Theorem 6*

$$L^2 - \lim_{t \rightarrow \infty} t^{(\beta - \beta^*)/\alpha} \sum_j X_j^\beta(t) = M_\infty \int_0^\infty x^{\beta - \beta^*} \rho(dx)$$

for $\Re \beta > \beta_a$.

Proof. Along the same line, the proof is reduced to showing that

$$\sup_{t \geq 0} t^{(\beta^* - \beta)/\alpha} y^{-\beta^*} \mathbb{E} \sum_j \left(X_j^{(y)}(t) \right)^\beta$$

is bounded uniformly in $y \in]0, 1[$. And the latter follows by noting that (5) and (15) imply

$$t^{(\beta - \beta^*)/\alpha} \mathbb{E} \sum_j \left(X_j^{(y)}(t) \right)^\beta \rightarrow y^{\beta^*} \int_0^\infty x^{\beta - \beta^*} \rho(dx).$$

□

4.2 Comparison with Branching Random Walks in continuous time

In this section, we compare the case of positive indices of self-similarity $\alpha > 0$ which we have considered so far, with the simpler *homogeneous* case when $\alpha = 0$.

So suppose here that $\alpha = 0$. The process $Z(t) = \{Z_j(t)\}$ with $Z_j(t) = -\log X_j(t)$ is then a continuous-time *branching random walk*, as studied in [36, 16]. We have in the notation of Section 3.1 that $\gamma(n, \beta) = \psi^n(\beta)$ and the identity (9) for the moments of power sums thus becomes $m(t, \beta) = \exp(-t\psi(\beta))$, a formula which is well-known in the context of branching random walk; see [36, 16].

It is read from the work of Biggins [15, 16] that for every $\beta > \beta_a$, the process

$$W(t, \beta) := \exp(t\psi(\beta)) \sum_j X_j^\beta(t), \quad t \geq 0$$

is a martingale with càdlàg paths, which converges almost surely and in their mean as $t \rightarrow \infty$ provided that $\psi(\beta) < \beta\psi'(\beta)$, see [11] for details. Note that this holds in the special case when $\beta = \beta^*$ is the Malthusian exponent, for which there is the identity $W(t, \beta^*) = M(t, \beta^*) = M_t$.

When furthermore the reproduction law is not arithmetic, the asymptotic behaviour of the empirical distribution of sizes can be described as follows: for every continuous function $f : \mathbb{R} \rightarrow \mathbb{R}$ with compact support,

$$\lim_{t \rightarrow \infty} \sqrt{t} e^{-t(\beta\psi'(\beta) - \psi(\beta))} \sum_j f(t\psi'(\beta) - Z_j(t)) = \frac{W(\infty, \beta)}{\sqrt{2\pi|\psi''(\beta)|}} \int_{-\infty}^{\infty} f(-z) e^{\beta z} dz. \quad (18)$$

where $W(\infty, \beta)$ is the terminal value of $W(t, \beta)$. See [36, 16].

Theorem 6 bears obvious similarities with (18). It is interesting to observe that in the homogeneous case $\alpha = 0$, sizes decay exponentially fast and the limiting scaled empirical measure is always exponential (up-to a random factor), whereas for $\alpha > 0$ the decay of sizes is power-like and the limiting scaled empirical measure depends crucially on the structural measure σ (more precisely, σ can be recovered from the limiting scaled empirical measure for $\alpha > 0$, but not for $\alpha = 0$).

5 Examples

5.1 Filippov's example revisited

Extending an example in Filippov (see [19], section 8) consider the structural measure

$$\sigma(dx) = \lambda x^{\theta-1} dx, \quad x \in]0, 1]$$

with parameters $\lambda > \min(\theta, 0)$ and *arbitrary* $\theta \in \mathbb{R}$. For $\lambda < \theta + 1$ the fragmentation is mean-value dissipative. We have

$$\phi(\beta) = \frac{\lambda}{\theta + \beta}, \quad \beta^* = \lambda - \theta, \quad \psi(\beta) = \frac{\beta - \beta^*}{\beta + \theta}.$$

The characteristic function is thus meromorphic in \mathbb{C} with a unique simple pole at $\beta_a = -\theta$.

Computing

$$\gamma(n, \beta) = \frac{(A)_n}{(B)_n}$$

we see that this is the ratio of two Pochhammer factorials, with $A = (\beta - \beta^*)/\alpha$ and $B = (\theta + \beta)/\alpha$, thus $m(t, \beta) = {}_1F_1(A; B; -t)$ is Kummer's hypergeometric function. The analytical extension of γ is

$$\gamma(z, \beta) = \frac{\Gamma(A+z)\Gamma(B)}{\Gamma(B+z)\Gamma(A)}.$$

Computing the moments we obtain

$$\int_0^\infty x^{\alpha k} \rho(dx) = (\lambda/\alpha)_k$$

which identifies ρ as

$$\rho(dx) = \frac{\alpha}{\Gamma(\lambda/\alpha)} x^{\lambda-1} e^{-x^\alpha} dx, \quad x \geq 0.$$

Note that the shape parameter θ cancels and does not appear in ρ . It follows that $\sigma_t := x^{-\beta^*} \sigma_t^*$ (the intensity of $M \delta_{t^{1/\alpha} X_j(t)}$) converges to the measure

$$x^{-\beta^*} \rho(dx) = \frac{\alpha}{\Gamma(\lambda/\alpha)} x^{\theta-1} e^{-x^\alpha} dx, \quad x \geq 0$$

in accord with the case $\lambda = 2, \theta = 1$ considered in [18] in connection with the conservative binary fragmentation with ξ_1 uniform and $\xi_2 = 1 - \xi_1$.

It follows that the mean number of particles satisfies

$$m(t, 0) \sim \frac{\Gamma(\theta/\alpha)}{\Gamma(\lambda/\alpha)} t^{(\lambda-\theta)/\alpha}, \quad t \rightarrow \infty.$$

which agrees with a special case in [19]. Of course, this formula makes sense only for $\theta > 0$, because our asymptotics for $m(t, \beta)$ hold only for $\Re \beta > \beta_a$, thus for $\theta \leq 0$ the value $\beta = 0$ is not considered.

The case $\lambda = 1, \theta = 0$, when $\sigma(dx) = x^{-1} dx$ corresponds to the conservative fragmentation generated by the uniform stick-breaking, as in the example in Section 2.1 (but without removing a piece). It is well known that the distribution of a size-biased pick from $\{\xi_j\}$ is uniform, and this implies that the intensity measure of $\sum \delta_{\xi_j}$ is indeed $\sigma(dx) = x^{-1} dx$.

5.2 Hypergeometrics

The following is a further generalisation of Filippov's example, and covers the class of dissipative binary fragmentations treated in [3]. Consider a Dirichlet polynomial

$$g(x) = \sum_{j=1}^p \lambda_j x^{\theta_j-1} \tag{19}$$

which is non-negative on $]0, 1]$ and has real parameters satisfying

$$\sum_{j=1}^p \frac{\lambda_j}{\theta_j} > 1.$$

Then $\sigma(dx) = g(x)dx$ is a measure on $]0, 1]$ with rational characteristic function

$$\phi(\beta) = \sum_{j=1}^p \frac{\lambda_j}{\theta_j + \beta}$$

and by the assumption the rightmost root of $\phi(\beta) = 1$ is positive, denote it also $\beta_1 = \beta^*$ and denote further roots β_2, \dots, β_p (the roots are certainly different from the poles of ϕ).

Observe that

$$\psi(\beta) = \prod_{j=1}^p \frac{\beta - \beta_j}{\beta + \theta_j},$$

thus assuming $\alpha = 1$ (without loss of generality) we have

$$m(t, \beta) = \sum_{n=0}^{\infty} \frac{(-t)^n}{n!} \prod_{j=1}^p \frac{(\beta - \beta_j)_n}{(\beta + \theta_j)_n}$$

where we recognise a generalised hypergeometric function of the type ${}_pF_p$. By (3) we have $m(t, \beta) \sim c(\beta)t^{\beta-\beta^*}$ for $\Re\beta > \beta_a$. Noting that

$$\psi'(\beta^*) = \frac{1}{\beta^* + \theta_1} \prod_{j=2}^p \frac{\beta^* - \beta_j}{\beta^* + \theta_j}$$

and manipulating infinite products, the coefficient is evaluated in terms of the gamma function as

$$c(\beta) = \prod_{j=2}^p \frac{\Gamma(\beta^* - \beta_j)}{\Gamma(\beta - \beta_j)} \prod_{j=1}^p \frac{\Gamma(\beta + \theta_j)}{\Gamma(\beta^* + \theta_j)}.$$

This allows to recover the density by Mellin inversion as

$$\frac{d\rho}{dx} = \frac{1}{2\pi i} \frac{\prod_{j=2}^p \Gamma(\beta^* - \beta_j)}{\prod_{j=1}^p \Gamma(\beta^* + \theta_j)} \int_{-i\infty}^{i\infty} \frac{\prod_{j=1}^p \Gamma(z + \beta^* + \theta_j)}{\prod_{j=2}^p \Gamma(z + \beta^* - \beta_j)} x^{-z-1} dz, \quad x \geq 0$$

which is an instance of Meijer's G -function, see [27].

The limit measure is uniquely determined by the integer moments which can be written as

$$\int_0^\infty x^k \rho(dx) = \frac{(k-1)!}{\psi'(\beta^*)} \prod_{j=1}^p \frac{(\beta^* + 1 + \theta_j)_{k-1}}{(\beta^* + 1 - \beta_j)_{k-1}},$$

where the derivative may be also computed as

$$\psi'(\beta^*) = \sum_{j=1}^p \frac{\lambda_j}{(\beta^* + \theta_j)^2}.$$

6 Fragmentations with infinite reproduction measure

We sketch how the preceding results can be extended to a class of self-similar conservative or dissipative fragmentations with infinite reproduction measure. Such processes were introduced in [7] where the reproduction law was called ‘dislocation measure’.

Let ν be a measure on the infinite simplex

$$\Delta = \{(s_j) : s_j \geq 0, s_j \downarrow 0, \sum_{j=1}^\infty s_j \leq 1\}$$

such that the integral

$$\psi(\beta) := \int_\Delta (1 - \sum s_j^\beta) \nu(ds) \tag{20}$$

satisfies $1 < \psi(\beta) < \infty$ for some $\beta > 0$. For $\alpha \geq 0$ we can define a fragmentation process $(X(t), t \geq 0)$ with the property that a generic particle of size x gives birth to a collection of particles of sizes xs_j with $s = (s_j) \in B$ at rate $x^\alpha \nu(B)$, where s_j runs over nonzero coordinates of s and $B \subset \Delta$ runs over Borel sets of finite ν -measure.

If ν is a probability measure it can be regarded as a reproduction law by defining (ξ_j) to be a random element of Δ with distribution ν . In this case the structural measure is identified with the superposition of marginal distributions of ν , and the definition (20) agrees with our definition of the characteristic function in Section 2.1. The case $\nu < \infty$ is easily reduced to the case $\nu(\Delta) = 1$ by the obvious time-change.

In the case $\nu(\Delta) = \infty$ some features of the fragmentation process are different, in particular, each particle produces infinitely many generations within arbitrarily small time period. As a consequence, the life-time of a particle is not a well-defined quantity, we do not have a tree representation for the genealogical structure, nor equation like (6). Still, we can define β_a and β^* exactly as in the case of finite measure, and consider β -sizes for $\Re\beta > \beta_a$. Formula (9) remains valid and can be proved by an argument

exploiting approximation of ν by suitable finite measures, or by using the methods developed in [10, 12]. For $\alpha > 0$ conclusions of Theorems 4 and 6 remain valid if we assume that

$$\psi'(\beta^*+) < \infty \quad \text{and} \quad \int_{\Delta} \left(\sum_{j=1}^{\infty} s_j^{\beta^*} - 1 \right)^2 \nu(ds) < \infty,$$

and impose a non-arithmeticity condition on ν . The analog of Corollary 2 is shown by arguments similar to those in Theorem 2 of [8], and the analog of distributional equation (4) generalises in the form of the Laplace transform identity

$$\mathcal{L}(\theta) = \int_{\Delta} \prod_{j=1}^{\infty} \mathcal{L}(\theta s_j^{\beta^*}) \nu(ds) \quad (21)$$

for $\mathcal{L}(\theta) := \mathbb{E} \exp(-\theta M_{\infty})$. The question about the uniqueness of solution to (21) with infinite measure seems to have not been considered before and remains open.

Finally, we observe the obvious interchangeability between parameters α and β (which holds no matter whether the reproduction measure is finite or infinite). In the variables $\tilde{\xi}_j = \xi_j^{\beta}$ the fragmentation process has the life-time parameter α/β and differs only by a particle-wise transformation of sizes. This observation enables us to handle situations when the measure ν is not supported by the infinite simplex Δ , but rather by

$$\Delta_{\beta} = \{(s_j) : s_j \geq 0, s_j \downarrow 0, \sum_{j=1}^{\infty} s_j^{\beta} \leq 1\}$$

for some $\beta > 0$. Note also that the change of variables with $\beta < 0$ yields mathematically equivalent (though physically curious) process where the individual ‘fragments’ grow, but the decaying life-times slow down the total increase of size.

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